Math. Z. 236, 113–131 (2001)
Digital Object Identifier (DOI) 10.1007/s002090000172

Mathematische Zeitschrift

Characteristic space-time estimates for the wave equation

Allan Greenleaf[1], **Gunther Uhlmann**[2]

[1] Department of Mathematics, University of Rochester, Rochester, NY 14627
(e-mail: allan@math.rochester.edu)
[2] Department of Mathematics, University of Washington, Seattle, WA 98195
(e-mail: gunther@math.washington.edu)

Received August 16, 1999; in final form November 25, 1999 /
Published online December 8, 2000 – © Springer-Verlag 2000

1. Introduction

Define an operator $U : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1})$ by

$$Uf(x,t,\omega) = \square^{-1}(f(x)\cdot\delta(t - x\cdot\omega)),$$

where $\square^{-1} : \mathcal{E}'(\mathbb{R}^{n+1}) \to \mathcal{D}'(\mathbb{R}^{n+1})$ denotes convolution with the forward fundamental solution of the d'Alembertian on $\mathbb{R}^{n+1}$. The main results of this paper concern the mapping properties of U on Sobolev and Besov spaces. The mapping properties of U on certain Fréchet spaces of distributions were previously studied by Melin [8,9].

The operator U arises naturally in the study of scattering by a potential on $\mathbb{R}^n$ [4,8,9,13]. If $\alpha(s,\theta,\omega)$ denotes the scattering kernel of a potential $q(x)$, so that $\delta(s-t)\delta(\theta-\omega) + \alpha(s-t,\theta,\omega)$ is the Schwartz kernel of the scattering operator for q, then an interesting problem is to determine to what extent the backscattering $\alpha(s,-\omega,\omega), \forall s \in \mathbb{R}, \omega \in \mathbb{S}^{n-1}$, determines q. Restricting formula (14) of [13] to $\theta = -\omega$, one finds that, for two potentials q_1, q_2 on $\mathbb{R}^3$ with scattering kernels α_1, α_2,

$$\begin{aligned}&(\alpha_1 - \alpha_2)(s,-\omega,\omega)\\ &\quad = -\frac{1}{8\pi^2}\int_{\mathbb{R}^3} \frac{\partial}{\partial s}(u_1 *_{\mathbb{R}} u_2)(x,-s,\omega)[q_1(x) - q_2(x)]dx,\end{aligned}$$

where u_1, u_2 are the solutions to the continuation problems

Research supported in part by grants from the National Science Foundation

$$(\Box + q_j(x))u_j(x,t,\omega) = 0 \quad on \quad \mathbb{R}^{3+1},$$
$$u_j(x,t,\omega) = \delta(t - x\cdot\omega), t << 0.$$

If one tries as in [4] to form a solution $u_j = \sum_{k=0}^{\infty} u_j^{(k)}$ with $u_j^{(0)} = \delta(t - x\cdot\omega)$ and $u_j^{(k+1)} = -\Box^{-1}(q_j(x)\cdot u_j^{(k)}), k \geq 0$, then the first two terms give $u_j(x,t,\omega) \simeq \delta(t-x\cdot\omega) - U(q_j)(x,t,\omega)$, with U the operator above, and one is naturally interested in controlling $U(q)$ with respect to various function space norms.

To put the operator U in context, note that if one replaces the characteristic hyperplane $\{t = x\cdot\omega\}$ with the noncharacteristic $\{t = 0\}$, and suppress the ω variables, then the corresponding operator

$$Vf(x,t) = \Box^{-1}(f(x)\cdot\delta(t)), \quad V : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{n+1}),$$

is by Duhamel's Principle the solution of the Cauchy problem

$$\Box Vf(x,t) = 0 \quad on \quad \mathbb{R}^{n+1}, \quad Vf(x,0) = 0, \partial_t Vf(x,0) = f(x),$$

and estimates for V include the familiar Strichartz estimates[14].

Let L^p_s denote the standard Sobolev spaces of distributions with s derivatives in L^p, and $B^s_{p,\infty}$ the standard Besov spaces of distributions with s derivatives having Littlewood-Paley components associated with large frequencies uniformly in L^p [7, v.3,p.472],[15]. L^p_s and $B^s_{p,\infty}$ are localizable and we work with their compactly supported and local variants, e.g., $L^p_{s,\text{comp}}$ and $B^s_{p,\infty,\text{loc}}$. If X is a smooth manifold of dimension N, and $\Lambda \subset T^*X\backslash 0$ is a conic Lagrangian submanifold, let $I^\mu(\Lambda)$ denote the Fourier integral distributions on X of order μ associated to Λ [7, v.4,p.4]. If Λ is a conormal bundle, $\Lambda = N^*Z$ with $Z \subset X$ of codimension k, then one has the continuous inclusions

$$I^\mu(\Lambda) \hookrightarrow B^m_{p,\infty,\text{loc}} \hookrightarrow L^p_{m-\epsilon,\text{loc}}, \quad \forall\epsilon > 0, \quad 1 < p < \infty,$$

for $m = -\mu - \frac{N}{4} + k(\frac{1}{2} - \frac{1}{p'})$, where $p' = \frac{p}{p-1}$ is the dual exponent to p.

One of the most interesting features of the operator U is the limited regularity that it possesses. This can already be seen by applying U to a test function $f \in C_0^\infty(\mathbb{R}^n)$. A calculation (see below) shows that near the characteristic hyperplane $\{t = x\cdot\omega\}$, one has $Uf(x,t,\omega) \simeq H(t - x\cdot\omega)$, where H is the Heaviside function. Thus, no matter how smooth f is, Uf is not smoother than a general element of

$$I^{-\frac{n+1}{2}}(N^*\{t - x\cdot\omega\}) \hookrightarrow B^{\frac{1}{p}}_{p,\infty,\text{loc}}(\mathbb{R}^{n+1}\times\mathbb{S}^{n-1}) \hookrightarrow L^p_{\frac{1}{p}-\epsilon},$$

(1.1) $\qquad \forall\epsilon > 0, \quad 2 \leq p < \infty.$

The operator U is a generalized Fourier integral operator (see [11,6]) associated with two intersecting canonical relations, $\Lambda_1, \Lambda_2 \subset (T^*\mathbb{R}^{n+1} \times \mathbb{S}^{n-1}\backslash 0) \times T^*\mathbb{R}^n$. Two complications arise: Λ_2 is degenerate, in that there are points in Λ_2 where $\pi_R : \Lambda_2 \to T^*\mathbb{R}^n$ is not a submersion, and both Λ_1 and Λ_2 have points which project into the zero-section, $0_{T^*\mathbb{R}^n}$. This second fact is reflected in the limited regularity of U. We are able to prove:

Theorem 1.
a) For $n \geq 2$*,*

$$U : L^2_{s,\text{comp}}(\mathbb{R}^n) \to L^2_{s+1,\text{loc}}(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1}), \quad \forall s < -\frac{1}{2},$$

with the endpoint result

$$U : L^2_{-\frac{1}{2},\text{comp}}(\mathbb{R}^n) \to B^{\frac{1}{2}}_{2,\infty,\text{loc}}(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1}).$$

b) For $n \geq 2$*,*

$$U : L^p_{s,\text{comp}}(\mathbb{R}^n) \to L^{p'}_{s+\frac{1}{p'},\text{loc}}(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1}),\ s < 0,\ \frac{4n}{2n+1} < p < 2.$$

Further estimates can be obtained by interpolating (*a*) and (*b*), but we will not state these or be concerned with the endpoint results in (*b*) because these estimates are probably not sharp. In fact, we conjecture that (*b*) holds for $\frac{2n}{n+1} < p < 2$. To see this, we write

$$Uf(x,t,\omega) = K_+ *_{\mathbb{R}^{n+1}} (f(x)\delta(t - x\cdot\omega)) \tag{1.2}$$

where $K_+(x,t) \simeq \partial_t^{\frac{n-3}{2}}(\frac{\delta(t-|x|)}{|x|^{\frac{n-1}{2}}})$ is the forward fundamental solution of $\square$. Thus,

$$Uf(x,t,\omega) \simeq \partial_t^{\frac{n-3}{2}}(v(x,t,\omega)), \tag{1.3}$$

where

$$v(x,t,\omega) = \frac{\delta(t-|x|)}{|x|^{\frac{n-1}{2}}} *_{\mathbb{R}^{n+1}} (f(x)\delta(t - x\cdot\omega)). \tag{1.4}$$

To calculate v, consider, for fixed $\omega \in \mathbb{S}^{n-1}$, the map $\Phi^\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined by $\Phi^\omega(y,z) = (y+z, y\cdot\omega + |z|) = (x,t)$. Then, $v(x,t,\omega)dxdtd\omega$ is the push-forward by Φ^ω of the measure $\frac{f(y)}{|z|^{\frac{n-1}{2}}}dydz$ on $\mathbb{R}^n \times \mathbb{R}^n$. By the coarea formula [2, Sect. 3.2],

$$v(x,t,\omega) = \int_{(\Phi^\omega)^{-1}(x,t)} \frac{f(y)}{|z|^{\frac{n-1}{2}}} \frac{d\nu^{(x,t)}_{n-1}(y,z)}{J_{n+1}(D\Phi^\omega(y,z))}, \tag{1.5}$$

where $d\nu_{n-1}^{(x,t)}$ is $(n-1)$-dimensional surface measure on $(\Phi^\omega)^{-1}(x,t)$ and $J_{n+1}(D\Phi^\omega(y,z))$ is (essentially) the maximal $(n+1)\times(n+1)$ minor of $D\Phi^\omega$. Since $D\Phi^\omega = \begin{bmatrix} I_n & I_n \\ \omega & \frac{z}{|z|} \end{bmatrix}$, we have $J_{n+1}(D\Phi^\omega(y,z)) \simeq ||\omega \wedge \frac{z}{|z|}|| + |\omega - \frac{z}{|z|}|$. Taking $\omega = e_1$ for simplicity, and writing $y=(y_1,y')$, we have that

$$(\Phi^\omega)^{-1}(x,t) = \Big\{(y,z) : y+z = x, y_1 + |z| = t\Big\}$$

is smooth and nonempty for $t > x_1$ and is parametrized (via $y \to (y, x-y)$) by the paraboloid

$$\Big\{y \in \mathbb{R}^n : y_1 = \frac{t+x_1}{2} - \frac{|y'-x'|^2}{2(t-x_1)}\Big\}.$$

Thus,

(1.6) $$v(x,t,e_1)$$

$$\simeq (t-x_1)^{\frac{n-5}{2}} \int f\Big(\frac{t+x_1}{2} - \frac{|y'-x'|^2}{2(t-x_1)}, y'\Big) \frac{dy'}{((t-x_1)^2 + |y'-x'|^2)^{\frac{n-3}{4}}}.$$

For $f \in C_0^\infty(\mathbb{R}^n)$, split the domain of integration in (1.6) into $\{|y'-x'| \le t - x_1\}$ and $\{t - x_1 \le |y'-x'| \le c(t-x_1)^{1/2}\}$. Taking into account the smooth dependence on ω, this yields $Uf \simeq H(t - x\cdot\omega) \in I^{-\frac{n+1}{2}}(N^*\{t = x\cdot\omega\})$, so that (1.1) and the comments above it hold.

To understand the restrictions on the $L^p_s \to L^q_r$ boundedness of U, fix $f \in C_0^\infty(\mathbb{R}^n)$, $f \ge 0$, and set $f_\delta(y) = f(\frac{y}{\delta})$. Then, for $s \le \frac{n}{p'}, p < \frac{n}{1-s}$, we have $||f_\delta||_{L^p_{-s}} \simeq \delta^{s+\frac{n}{p}}$ and we ask whether it is possible that $||Uf_\delta||_{L^q_r} \le C||f_\delta||_{L^p_s}$ uniformly as $\delta \to 0^+$; for simplicity we take $n=3$ and $r=0$ and ask for which p, q and s one has local boundedness $U : L^p_{-s} \to L^q$. From (1.6), one can see that for $\delta^2 < t - x_1 < \delta$, $\delta < |x'| < (t-x_1)^{\frac{1}{2}}$, and $\Big||x'| - (\frac{x_1}{t-x_1})^{\frac{1}{2}}\Big| < \delta$ we have $Uf_\delta \ge c\frac{\delta^2}{|x'|}$. Thus,

$$\int\int |Uf_\delta|^q dxdt \ge c \int_{\delta^2}^{\delta} \int_{\delta}^{(t-x_1)^{\frac{1}{2}}} \frac{\delta^{2q}}{\rho^q} \cdot \frac{\rho}{t-x_1} \cdot \rho\delta d\rho d(t-x_1) \Longrightarrow$$

$||Uf_\delta||_{L^q} \ge c\delta^{\frac{5}{2q}+\frac{3}{2}}$, $q<3$; $\ge c\delta^{\frac{7}{q}}|ln(\delta)|^{\frac{2}{q}}$, $q=3$; and $\ge c\delta^{\frac{q+4}{q}}$, $q>3$. Thus, if U is bounded from $L^p_{-s}(\mathbb{R}^3) \to L^q(\mathbb{R}^4 \times \mathbb{S}^2)$ locally, then $s \le 3 + \frac{5}{2q} - \frac{3}{p}, q < 3$; $\quad s \le \frac{7}{9} - \frac{3}{p}, \quad q = 3$ and $s \le 1 + \frac{4}{q} - \frac{3}{p}, \quad q > 3$. If, in the context of Theorem 1, one takes $q = p', s = \frac{1}{p'}$, then necessarily $p > \frac{3}{2}$.

Notation: We use the notation $<\xi> = (1+|\xi|^2)^{\frac{1}{2}}$ for $\xi \in \mathbb{R}^n$ and in a product setting, $<\xi,\eta>$ denotes $<(\xi,\eta)> = (1+|\xi|^2+|\eta|^2)^{\frac{1}{2}}$. The Euclidian innner product is denoted by $\xi \cdot \eta$.

2. Microlocal structure and estimates for U

Let $K_+ \in \mathcal{D}'(\mathbb{R}^{n+1})$ be the forward fundamental solution of the wave equation. Then the Schwartz kernel of the operator U is

$$K_U(x,t,\omega,y) = \int_{\mathbb{R}} K_+(x-y,t-s)\delta(s-y\cdot\omega)ds. \tag{2.1}$$

We may represent K_+ by

$$K_+(x,t) = \int_{\mathbb{R}^{n+2}} e^{i(x\cdot\xi+t\tau+\frac{\sigma}{2\tau^2}(\tau^2-|\xi|^2))} b(x,t;(\xi,\tau),\sigma)d\xi d\tau d\sigma,$$

with $b(x,t;(\xi,\tau),\sigma)$ belonging to the product-type symbol class $S^{-2,0}$ $(\mathbb{R}^{n+1}\times(\mathbb{R}^{n+1}\backslash 0)\times\mathbb{R})$ (cf.,[11,6]), where

$$S^{M,M'}(\mathbb{R}^N_x\times(\mathbb{R}^m_\theta\backslash 0)\times\mathbb{R}^n_\sigma) =$$

$$\Big\{a(x,\theta,\sigma) : |\partial_x^\gamma\partial_\theta^\beta\partial_\sigma^\alpha a(x,\theta,\sigma)| \le C_{\alpha\beta\gamma K} < \theta,\sigma >^{M-|\beta|} < \sigma >^{M'-|\alpha|}, x\in K \subset\subset \mathbb{R}^N\Big\}. \tag{2.2}$$

Writing $\delta(s-y\cdot\omega) = \int_{\mathbb{R}} e^{i(s-y\cdot\omega)\rho}c(s,\rho)d\rho, c \in S^0(\mathbb{R}\times(\mathbb{R}\backslash 0))$ compactly supported in $s-y\cdot\omega$, we obtain the oscillatory representation

$$K_U(x,t,\omega,y) = \int_{\mathbb{R}^{n+3}} e^{i((x-y)\cdot\xi+(t-s)\tau+(s-y\cdot\omega)\rho+\frac{\sigma}{2\tau^2}(\tau^2-|\xi|^2))} b(x,t,\omega,y;(\xi,\tau),\sigma)d\xi d\tau d\sigma d\rho ds.$$

Since the s-derivative of the phase is $\rho-\tau$, we may integrate by parts in s and obtain a gain of $(1+|\rho-\tau|)^{-N}, \forall N$, and then integrating in ρ and s yields

$$K_U(x,t,\omega,y) = \int_{\mathbb{R}^{n+2}} e^{i((x-y)\cdot\xi+(t-y\cdot\omega)\tau+\frac{\sigma}{2\tau^2}(\tau^2-|\xi|^2))} a(x,t,\omega,y;(\xi,\tau),\sigma)d\xi d\tau d\sigma, \tag{2.3}$$

with $a\in S^{-2,0}((\mathbb{R}^{n+1}\times\mathbb{S}^{n-1}\times\mathbb{R})\times(\mathbb{R}^{n+1}\backslash 0)\times\mathbb{R})$. Standard wave-front set analysis shows that

$$WF(K_U) \subset \Lambda_1\cup\Lambda_2,$$

where

$$\begin{aligned}\Lambda_1 &= N^*\{(x,t,\omega,y) : x=y, t=x\cdot\omega\}\\ &= \{(x,x\cdot\omega,\omega,\xi,\tau,-\tau i^*_\omega x;x,-\xi-\tau\omega)\\ &\qquad : x\in\mathbb{R}^n,\omega\in\mathbb{S}^{n-1},(\xi,\tau)\in\mathbb{R}^{n+1}\backslash 0\},\end{aligned} \tag{2.4}$$

(where, for $\omega \in \mathbb{S}^{n-1}$, $i_\omega : T_\omega \mathbb{S}^{n-1} \to T_\omega \mathbb{R}^n$ is the inclusion map) and

$$\begin{aligned} \Lambda_2 &= N^*\{|t - y \cdot \omega|^2 = |x-y|^2\} \\ &= \{(x, x\cdot\omega + r(1+\omega\cdot\nu), \omega, \tau\nu, \tau, -\tau i_\omega^*(x+r\nu); x+r\nu, -\tau(\nu+\omega)) \\ &\qquad : x \in \mathbb{R}^n, \omega, \nu \in \mathbb{S}^{n-1}, r \in \mathbb{R}, \tau \in \mathbb{R}\backslash 0\}. \end{aligned} \tag{2.5}$$

Λ_1 and Λ_2 are smooth, conic lagrangian submanifolds of $T^*(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1} \times \mathbb{R}^n)\backslash 0$ that intersect cleanly in codimension 1, and the phase function of (2.3) is a "multiphase" function parametrizing the pair (Λ_1, Λ_2) in the sense of [12]. Since $a \in S^{-2,0}$, we have

$$U \in I^{-(\frac{n+4}{4}), -\frac{1}{2}}(\Lambda_1, \Lambda_2),$$

in the notation of [11,6]. Microlocally, away from $\Lambda_1 \cap \Lambda_2$, we have $U \in I^{-\frac{n+6}{4}}(\Lambda_1 \backslash \Lambda_2)$ and $U \in I^{-\frac{n+4}{4}}(\Lambda_2 \backslash \Lambda_1)$. Using a parabolic cutoff, introduced in this context by Melrose [10](cf., [3]), we can actually decompose

$$U \in I_{\frac{1}{2}}^{-\frac{n+4}{4}}(\Lambda_1) + I_{\frac{1}{2}}^{-\frac{n+4}{4}}(\Lambda_2), \tag{2.6}$$

where the subscript $\frac{1}{2}$ refers to amplitudes which lie in the symbol class $S_{\frac{1}{2},\frac{1}{2}}$. In fact, let $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ near 0, and write the amplitude $a(x, t, \omega; (\xi, \tau), \sigma)$ in (2.3) as

$$\begin{aligned} a &= \chi(\frac{\sigma^2}{< \xi, \tau >}) \cdot a + (1-\chi)(\frac{\sigma^2}{< \xi, \tau >}) \cdot a \\ &= a_1 + a_2. \end{aligned}$$

Then, in the oscillatory integral (2.3) corresponding to a_1, we first integrate in σ; since the region being integrated over is contained in $\{|\sigma| \le c < \xi, \tau >^{1/2}\}$, the resulting expression is

$$K_{U_1}(x, t, \omega, y) = \int_{\mathbb{R}^{n+1}} e^{i((x-y)\cdot\xi + (t - y\cdot\omega)\tau)} b_1(x, t, \omega, y; \xi, \tau) d\xi d\tau \tag{2.7}$$

with $b_1 \in S_{\frac{1}{2}}^{-\frac{3}{2}}(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1} \times (\mathbb{R}^{n+1}\backslash 0))$, yielding $U_1 \in I_{\frac{1}{2}}^{-\frac{n+4}{4}}(\Lambda_1)$. On the support of a_2, we have $|\sigma| \ge < \xi, \tau >^{1/2}$, so that the product-type estimates (2.2) (with $M = -2, M' = 0$) imply $a_2 \in S_{\frac{1}{2},\frac{1}{2}}^{-2}(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1} \times (\mathbb{R}^{n+2}\backslash 0))$, whence the corresponding operator, U_2, belongs to $I_{\frac{1}{2}}^{-\frac{n+4}{4}}(\Lambda_2)$. We note for future use that the amplitude a_2 actually gains $< \xi, \tau >^{-1}$ for each differentiation in ξ or τ ; we will refer to this as type $(1,0), (\frac{1}{2}, 0)$.

Furthermore, we can assume that $|\xi| \simeq |\tau|$ on supp (a_2), since away from $\{|\xi| = |\tau|\}$, we have $K_U \in \Psi^{-2} \circ I^{-\frac{n-2}{4}}(\Lambda_1) \hookrightarrow I^{-\frac{n+6}{4}}(\Lambda_1)$.

To prove part (a) of Theorem B, it suffices to prove the desired regularity for each U_j. As mentioned in §1, there is a limitation on the regularity that Uf can possess, regardless of how smooth f is, due to the fact that each $\Lambda_j \subset (T^*(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1}) \backslash 0) \times T^*\mathbb{R}^n$ has points sitting over the zero-section, $0_{T^*\mathbb{R}^n}$. In the parametrization (2.4), the points where $\xi = -\tau\omega$ project to $0_{T^*\mathbb{R}^n}$, as do the points with $\nu = -\omega$ in (2.5). To understand the implications of this for the estimates, consider the following model, to which Λ_1 can be conjugated by a local diffeomorphism from $\mathbb{R}^{n+1} \times \mathbb{S}^{n-1}$ to $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n-1}$. Let $N, k \geq 1, l \geq 0$ be integers. With coordinates $y' \in \mathbb{R}^N$ and $x = (x', x'', x''') \in \mathbb{R}^N \times \mathbb{R}^k \times \mathbb{R}^l$, consider the canonical relation

$$\hat{\Lambda} = \Big\{(x', 0, x''', \xi', \xi'', 0; x', \xi') : (x', x''') \in \mathbb{R}^{N+l}, (\xi', \xi'') \in \mathbb{R}^{N+k} \backslash 0\Big\}$$

$$= N^* \Big\{(x', x'', x''', y') : x' = y', x'' = 0\Big\}. \tag{2.8}$$

Note that $\hat{\Lambda}$ is nondegenerate in that the projection $\pi_L : \hat{\Lambda} \to T^*\mathbb{R}^{N+k+l} \backslash 0$ is an embedding and $\pi_R : \hat{\Lambda} \to T^*\mathbb{R}^N$ is a submersion. Also, there are points in $\hat{\Lambda}$ sitting above the zero section, $0_{T^*\mathbb{R}^n}$, namely $\{\xi' = 0\}$, and

$$\begin{aligned}\pi_L \pi_R^{-1}(0_{T^*\mathbb{R}^n}) &= \{(x', 0, x''', 0, \xi'', 0) : (x', x''') \in \mathbb{R}^{N+l}, \xi'' \in \mathbb{R}^k \backslash 0\} \\ &= N^*\{x'' = 0\}.\end{aligned}$$

The operator $Tf(x) = f(x')\delta(x'')$ belongs to $I^{\frac{k-l}{4}}(\hat{\Lambda})$ and, as with the operator U, it fails to map $C_0^\infty(\mathbb{R}^N)$ to $C^\infty(\mathbb{R}^{N+k+l})$; indeed, for $f \in C_0^\infty(\mathbb{R}^N)$, $Tf(x)$ is a smooth multiple of $\delta(x'')$, which belongs to

$$I^{-\frac{N-k+l}{4}}(N^*\{x'' = 0\}) \hookrightarrow B^{-\frac{k}{2}}_{2,\infty,\mathrm{loc}}(\mathbb{R}^{N+k+l}) \hookrightarrow L^2_{-\frac{k}{2}-\epsilon}(\mathbb{R}^{N+k+l})$$

for any $\epsilon > 0$, and no better. In general, to obtain the boundedness properties on L^2-based Sobolev spaces that one expects from the nondegeneracy of the projections π_L, π_R, an operator in $I(\hat{\Lambda})$ must have a sufficiently negative order.

This behavior can be considered in a superficially more general setting. Let X, Y be manifolds with $dim(X) = M > dim(Y) = N$. Let $C \subset (T^*X \times T^*Y) \backslash 0_{T^*(X \times Y)}$ be a nondegenerate canonical relation, so that at all points $\pi_L : C \to T^*X$ is an immersion and $\pi_R : C \to T^*Y$ is a submersion. Since the zero section 0_{T^*Y} is lagrangian in T^*Y, by [5] we have that $\Gamma_C = \pi_l \pi_R^{-1}(0_{T^*Y}) \hookrightarrow T^*X$ is an immersed lagrangian in $T^*X \backslash 0$. If $\pi_X : T^*X \to X$ is the projection onto the spatial variable and we make the (generic) assumption that $rank(d(\pi_X|_{\Gamma_C}))$ is constant on an immersed neighborhood of $c_0 \in C$, say of constant rank $N+l$, then microlocally $\Gamma_C = N^*\gamma$, with $\gamma \hookrightarrow X$ a submanifold of dimension $N+l$. Letting $k = M - N - l = codim(\gamma)$, we may then introduce local coordinates $y' \in \mathbb{R}^N$ on Y and $x = (x', x'', x''') \in \mathbb{R}^N \times \mathbb{R}^k \times \mathbb{R}^l$ on X so that $C = \hat{\Lambda}$ as in (2.8). If $f \in C_0^\infty(Y)$ and $A \in I^*(C)$, then $Af \in I^*(\Gamma_C)$ is in general not smooth.

Returning to the model case, let $\Psi_\rho^m(\mathbb{R}^N)$ denote the class of pseudo-differential operators of order m and type $(\rho, 1-\rho)$ on $\mathbb{R}^N$.

Proposition 2.1. *Let* $A_j \in I_\rho^{\mu_j - \frac{l+k}{4}}(\hat{\Lambda})$, $j = 1, 2$, *for some* $\frac{1}{2} \le \rho \le 1$, *be properly supported. Then, if* $\mu_j < -\frac{k}{2}$, $j = 1, 2$, *we have*

$$A_2^* A_1 \in \Psi_\rho^{\mu_1+\mu_2}(\mathbb{R}^N).$$

Proof. We write

$$A_j f(x) = \int\limits_{\mathbb{R}^N \times (\mathbb{R}^{N+k} \setminus 0)} e^{i[(x'-y')\cdot\xi' + x''\cdot\xi'']} a_j(x, y, (\xi', \xi'')) f(y) dy d\xi' d\xi'',$$

with $a_j \in S_\rho^{\mu_j - \frac{k}{2}}(\mathbb{R}^{2N+k+l} \times (\mathbb{R}^{N+k} \setminus 0))$. Then the Schwartz kernel of $A_2^* A_1$ is

$$K_{A_2^* A_1}(z', y') = \int e^{i[(x'-y')\cdot\xi' - (x'-z')\cdot\eta' + x''\cdot(\xi''-\eta'')]} a_1 \overline{a}_2 dx d\xi' d\xi'' d\eta' d\eta''.$$

We may assume that supp $(a_j) \subset \{|\xi'| \le \frac{1}{2}|\xi''|\}$, since away from $\xi' = 0$ the projection π_R avoids $0_{T^*\mathbb{R}^n}$ and the desired result follows from the standard FIO calculus. The x-gradient of the phase in (2.9) is $(\xi' - \eta', \xi'' - \eta'')$, so we can integrate by parts in the regions $\{|\xi'| \ge \frac{3}{2}|\eta'|\}$ and $\{|\xi'| \le \frac{2}{3}|\eta'|\}$ and obtain a C^∞ kernel. Similarly, the regions $\{|\xi''| \ge \frac{3}{2}|\eta''|\}$ and $\{|\xi''| \le \frac{2}{3}|\eta''|\}$ both contribute C^∞ kernels. Thus, we can localize the amplitude to $\{|\xi'| \sim |\eta'|, |\xi''| \sim |\eta''|\}$ by multiplying by cutoff functions $\psi(\frac{<\xi'>}{<\eta'>})\psi(\frac{<\xi''>}{<\eta''>})$ where $\psi \in C_0^\infty(\mathbb{R}), \psi \equiv 0$ near 0 and $\psi \equiv 1$ on $[\frac{1}{2}, 2]$. We thus have been reduced to considering the modified kernel

$$\tilde{K}(z', y') = \int e^{i[(x'-y')\cdot\xi' - (x'-z')\cdot\eta']}$$

$$b(z', y', x, \xi', \eta')\psi(\frac{<\xi'>}{<\eta'>}) dx' dx''' d\xi' d\eta', \tag{2.10}$$

where

$$b(z',y',x,\xi',\eta')=\int e^{ix''\cdot(\xi''-\eta'')}\psi(\frac{<\xi''>}{<\eta''>})$$
$$a_1(x,y',\xi',\xi'')\overline{a}_2(x,z',\eta',\eta'')dx''d\xi''d\eta''$$
$$(2.11)\qquad =\int e^{ix''\cdot\xi''}a_1\tilde{*}_{\xi''}\overline{a}_2(x,y',z',\xi',\eta',\xi'')dx''d\xi'',$$

where $a_1\tilde{*}_{\xi''}\overline{a}_2$ is the modified partial convolution in ξ'' defined by

$$a_1\tilde{*}_{\xi''}\overline{a}_2(x,y',z';\xi',\eta';\xi'')$$
$$(2.12)\quad =\int a_1(x,y',\xi',\xi''+\eta'')\overline{a}_2(x,z',\eta',\eta'')\psi(\frac{<\xi''+\eta''>}{<\eta''>})d\eta''.$$

We will apply the following result, with $m_j=\mu_j-\frac{k}{2}, j=1,2$.

Proposition 2.2. *Let* $a_j\in S^{m_j}_\rho(\mathbb{R}^{N+k+l}\times(\mathbb{R}^{N+k}\backslash 0))$ *be supported in* $\{|\xi'|\le c|\xi''|\}$, $j=1,2$. *Then, if* $m_j<-k, j=1,2$ *, for any* $c_0>0$ *we have*

a)

$$a_1\tilde{*}_{\xi''}\overline{a}_2\in S^{m_1+m_2+k,0}_\rho(\mathbb{R}^{3N+k+l}\times(\mathbb{R}^k\backslash 0)_{\xi''}\times\mathbb{R}^{2N}_{\xi',\eta'})$$

in the region $\{<\xi''>\ge c_0<\xi',\eta'>\}$ *; and*

b)

$$a_1\tilde{*}_{\xi''}\overline{a}_2\in S^{m_1+m_2+k}_\rho(\mathbb{R}^{3N+k+l}\times(\mathbb{R}^{2N+k}\backslash 0))$$

in the region $\{<\xi''>\le c_0<\xi',\eta'>\}$.

Proof. For (a), we note that $\frac{1}{3}<\xi''>\le<\eta''>$ on the support of $\psi(\frac{<\xi''+\eta''>}{<\eta''>})$ and decompose the integral in (2.12) into two terms, corresponding to the regions $\{<\eta''>\ge 3<\xi''>\}$ and $\{\frac{1}{3}<\xi''>\le<\eta''>\le 3<\xi''>\}$. To estimate the size of $a_1\tilde{*}_{\xi''}\overline{a}_2$, we note that on the first region, the integrand is $\le c<\eta''>^{m_1+m_2}$, hence (by the elementary Proposition 2.3(a) below), the contribution to the integral is $\le c<\xi''>^{m_1+m_2+k}$, since $m_1+m_2<-k$. Differentiating in ξ'' brings a gain of $<\xi''>^{-1}$ to the integrand, while differentiating in ξ',η' brings no gain. On the support of the integrand of the second term, which has volume $\le c<\xi''>^k$, both $<\eta''>$ and $<\xi''+\eta''>$ are $\sim<\xi''>$, and so again the integral is $\le c<\xi''>^{m_1+m_2+k}$, and the same comments concerning derivatives apply. Hence, on $\{<\xi''>\ge c<\xi',\eta'>\}$, $a_1\tilde{*}_{\xi''}\overline{a}_2\in S^{m_1+m_2+k,0}_\rho(\mathbb{R}^{3N+k+l}\times(\mathbb{R}^k\backslash 0)\times\mathbb{R}^{2N})$.

On the other hand, if $< \xi'' > \leq c_0 < \xi' >$, then we simply have

$$|a_1 \tilde{*}_{\xi''} \overline{a}_2| \leq c \int_{<\eta''> \geq c < \xi'>} < \eta'' >^{m_1+m_2} d\eta'' \leq c < \xi' >^{m_1+m_2+k},$$

and differentiation in any of ξ', η', ξ'' produces a gain of $< \xi' >^{-1}$. This proves (b). □

Proposition 2.3. *(a) If a function $a(\xi', \xi'')$ on $\mathbb{R}^{n-k} \times \mathbb{R}^k$ satisfies*

$$|a(\xi', \xi'')| \leq c_1 < \xi', \xi'' >^m, \quad some \quad m < -k,$$

then $b(\xi')$ on $\mathbb{R}^{n-k}$ defined by

$$b(\xi') = \int_{\mathbb{R}^k} |a(\xi', \xi'')| d\xi''$$

satisfies $|b(\xi')| \leq c_2 < \xi' >^{m+k}$, with c_2 only depending on c_1, k, m, n.

(b) If $a \in S^m_\rho(\mathbb{R}^{n-k}_{\xi'} \times \mathbb{R}^k_{\xi''})$, $\quad m < -k$, and one sets

$$b(\xi'; x'') = \int_{\mathbb{R}^k} e^{ix'' \cdot \xi''} a(\xi', \xi'') d\xi'',$$

then, for each ξ', $b \in I^m_\rho(\{x'' = 0\})$, i.e., is conormal of order m for $\{x'' = 0\}$ and satisfies $|b(\xi'; x'')| \leq c_2 < \xi' >^{m+k}$.

Applying Proposition 2.2 as indicated, and noting that $m_j < -k$ if $\mu_j < -\frac{k}{2}$, we return to (2.11). The contribution from $\{(x'', \xi'') :< \xi'' > \geq c < \xi', \eta' >\}$, after integration by parts M times in x'', is $\leq c < \xi', \eta' >^{\mu_1+\mu_2+k-M}$. The contribution from $\{< \xi'' > \leq < \xi', \eta' >\}$ is also estimated via this integration by parts: the integrand is $\leq c_M < \xi'' >^{-M} < \xi', \eta' >^{\mu_1+\mu_2}$; upon integrating over the ball $\{< \xi'' > \leq c < \xi', \eta' >\}$, this is $\leq c < \xi', \eta' >^{\mu_1+\mu_2}$ (for $M > k$.) Derivatives of b are handled similarly, and thus $b \in S^{\mu_1+\mu_2}_\rho$. In (2.10), we may now perform stationary phase in x', η' to obtain

$$\tilde{K}(z', y') = \int e^{i(z'-y') \cdot \xi'} c(x'''; z', y'; \xi') d\xi' dx''' \tag{2.13}$$

with $c \in S^{\mu_1+\mu_2}_\rho$; integrating in x''', we find that $A_2^* A_1$ is a pseudo-differential operator on $\mathbb{R}^N$ with symbol of type $(\rho, 1-\rho)$ and order $\mu_1 + \mu_2$, finishing the proof of Proposition 2.1.

Using Proposition 2.1 with $A_1 = A_2 = U_1, N = n, k = 1, l = n - 1$ and $\mu_j = -1, j = 1, 2$, we have $U_1^* U_1 \in \Psi^{-2}(\mathbb{R}^n) \implies$

$$U_1 : L^2_{-1,\text{comp}}(\mathbb{R}^n) \to L^2_{\text{loc}}(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1}).$$

We will show that

$$(2.14)\qquad \begin{array}{l} U_1 : L^2_{s,\text{comp}} \to L^2_{s+1,\text{loc}}, \quad \forall s < 0, \quad and \\ U_1 : L^2_{\text{comp}} \to B^1_{2,\infty,\text{loc}} \hookrightarrow L^2_{1-\epsilon,\text{loc}}, \quad \forall \epsilon > 0, \end{array}$$

which will follow from the fact that, for all $r \in \mathbb{R}$, the Schwartz kernel
(2.15)

$$K_{<D>^r U_1 <D>^{-r}} \in I_{\frac{1}{2}}^{-\frac{n+4}{4}}(\Lambda_1) + C^\infty(\mathbb{R}^n, I^{-\frac{n+2}{2}+r}(N^*\{t - x\cdot\omega = 0\})),$$

where $< D >= (I + |D|^2)^{\frac{1}{2}}$. In fact, (2.15) implies that

$$\begin{aligned} &< D >^r U_1 < D >^{-r}: L^2_{-1,\text{comp}}(\mathbb{R}^n) \to L^2_{\text{loc}}(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1}) \\ &\quad + I^{-\frac{n+2}{2}+r}(N^*\{t - x\cdot\omega = 0\}) \\ &\Longrightarrow \\ &U_1 : L^2_{r-1,\text{comp}} \to L^2_{r,\text{loc}} + I^{-\frac{n+2}{2}}(N^*\{t - x\cdot\omega = 0\}) \\ &\hookrightarrow L^2_{r,\text{loc}} + B^1_{2,\infty,\text{loc}} \end{aligned}$$

since $I^\nu_\rho(N^*\{t = x\cdot\omega\}) \hookrightarrow B^{-\nu-\frac{n}{2}}_{2,\infty,\text{loc}}(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1})$. This implies (2.14).

To see (2.15), we work in the generality of Proposition 2.1 :

Proposition 2.4. *Let $\hat{\Lambda}$ be as in Proposition 2.1. Let $A \in I^{\mu-\frac{l+k}{4}}_\rho(\hat{\Lambda})$ and $B \in \Psi^{r_1}_\rho(\mathbb{R}^N), C \in \Psi^{r_2}_\rho(\mathbb{R}^{N+k+l})$ be properly supported. Then the Schwartz kernel*

$$K_{CAB} \in I^{\mu+r_1+r_2-\frac{l+k}{4}}_\rho(\hat{\Lambda}) + C^\infty(\mathbb{R}^N; I^{\mu+r_2-\frac{N+k+l}{4}}_\rho(N^*\{x'' = 0\})).$$

Proof. We start with the case $C = I \in \Psi^0$. We have the oscillatory representation

$$K_{AB}(x, z') = \int e^{i[(x'-y')\cdot\xi' + x''\cdot\xi'' + (y'-z')\cdot\eta']} a(*; \xi', \xi'') b(*; \eta') dy' d\xi' d\xi'' d\eta'$$

with $a \in S^{\mu-\frac{k}{2}}_\rho, \quad b \in S^{r_1}_\rho$. (Here, and throughout, we use $*$ to denote spatial variables irrelvant to the argument.) As in the proof of Proposition 2.1, we can assume that $supp(a) \subset \{|\xi'| \le \frac{1}{2}|\xi''|\}$, since outside of that region there is no zero-section problem and the standard result on the composition of FIOs with ΨDOs applies. Since the y'-gradient of the phase is $-\xi' + \eta'$, the standard integration by parts in y' shows that the contribution to K_{AB} from $\{< \eta' >\ge \frac{3}{2} < \xi' > + \frac{1}{2} < \xi'' >\}$ is in $C^\infty(\mathbb{R}^{N+k+l} \times \mathbb{R}^N)$. The remaining regions are $\{\frac{2}{3} < \frac{<\xi'>}{<\eta'>} < \frac{3}{2}\}$, $\{\frac{1}{2} < \xi'' >\ge< \xi' >\ge \frac{3}{2} < \eta' >\}$ and $\{\frac{1}{2} < \xi'' >\ge< \eta' >\ge \frac{3}{2} < \xi' >\}$. On the first region, stationary phase in y', η' yields an element of $I^{\mu+r_1-\frac{l+k}{4}}(\hat{\Lambda})$. On the second region, integration by parts M times in y' , followed by integration in η' over $\{<\eta'>\leq \frac{2}{3}<\xi'>\}$ yields

$$\int e^{i[(x'-y')\cdot\xi'+x''\cdot\xi'']}c(*;\xi'';\xi')dy'd\xi''d\xi',$$

with $c \in S^{\mu-\frac{k}{2},-\tilde{M}}(\mathbb{R}^{N+k+l}\times\mathbb{R}^N\times(\mathbb{R}^k\backslash 0)\times\mathbb{R}^N)$, where $-\tilde{M} = \mu+\frac{k}{2}+r_1+N-M(1-\rho)$. Integrating in y', ξ' when $-\tilde{M}<-N$ yields a kernel in

$$C^0(\mathbb{R}^N; I_\rho^{\mu-\frac{N+k+l}{4}}(N^*\{x''=0\})).$$

Derivatives in z' bring down powers of η' , which are handled by increasing M. Thus, the contribution to K_{AB} from the second region is in

$$C^\infty(\mathbb{R}^N; I_\rho^{\mu-\frac{N+k+l}{4}}(N^*\{x''=0\})).$$

Similarly, for the third region, we integrate by parts M times in y', integrate over
$\{<\xi'>\leq \frac{2}{3}<\eta'>\}$, and repeat the above argument to obtain a contribution in the same class. This finishes the proof when $C=I$. For general C, we note that since the projection $\pi_L : \hat{\Lambda}\to T^*\mathbb{R}^{N+k+l}\backslash 0$ avoids the zero section, the usual composition calculus of ΨDOs and FIOs applies. Furthermore, we have

$$\begin{aligned}\Psi_\rho^{r_2}(\mathbb{R}^{N+k+l}) \quad\circ\quad & C^\infty(\mathbb{R}^N; I_\rho^{\mu-\frac{N+k+l}{4}}(N^*\{x''=0\}))\\ &\subset C^\infty(\mathbb{R}^N; I_\rho^{\mu+r_2-\frac{N+k+l}{4}}(N^*\{x''=0\})),\end{aligned}$$

yielding the desired result. □

For $A=U_1$, we have $\mu=-1$ and $N+k+l=2n$, so Proposition 2.4 implies (2.15). This finishes the proof of the L^2 estimates, Thm. 1(a), for U_1. In fact, (2.14) allows one to go $\frac{1}{2}$ derivative higher; the limitation $s<-\frac{1}{2}$ in Thm. 1(a) comes from U_2, which we now turn to.

We first show that if $A\in I^{\mu-\frac{n}{4}}(\Lambda_2)$ is properly supported, then $A^*A\in \Psi_{0,0}^{2\mu}(\mathbb{R}^n)$ if $-\frac{5}{4}<\mu<0$. Since $\Psi_{1,0}^{-\mu}\circ\Psi_{0,0}^{2\mu}\circ\Psi_{1,0}^{-\mu}\subset\Psi_{0,0}^0$, by the Calderón-Vaillancourt theorem for $\Psi_{0,0}^0$, we have $A^*A : L^2_{\mu,\text{comp}}\to L^2_{-\mu,\text{loc}}$, and thus $A: L^2_{\mu,\text{comp}}\to L^2_{\text{loc}}$. From (2.3), we have

$$\begin{aligned}K_A(x,t,\omega,y) = \int_{\mathbb{R}^{n+2}} & e^{i((x-y)\cdot\xi+(t-y\cdot\omega)\tau+\frac{\sigma}{2\tau^2}(\tau^2-|\xi|^2))}\\ & a_{\mu-1}(x,t,\omega,y;(\xi,\tau),\sigma)d\xi d\tau d\sigma,\end{aligned} \tag{2.16}$$

with $a_{\mu-1} \in S^{\mu-1}_{\frac{1}{2}}$ supported in $\{c < \xi, \tau >^{\frac{1}{2}} \leq |\sigma| \leq c < \xi, \tau >\}$ Thus,

$$K_{A^*A}(z,y) = \int\int\int e^{i\phi(y,z,x,t,\omega,\xi,\sigma,\tau,\tilde{\xi},\tilde{\sigma},\tilde{\tau})}\overline{a_{\mu-1}}a_{\mu-1}dxdtd\omega d\xi d\sigma d\tau d\tilde{\xi}d\tilde{\sigma}d\tilde{\tau},$$

where the integral is over $\mathbb{R}^{n+1} \times \mathbb{S}^{n-1} \times \mathbb{R}^{n+2} \times \mathbb{R}^{n+2}$ and

$$\phi(y,z,x,t,\omega,\xi,\sigma,\tau,\tilde{\xi},\tilde{\sigma},\tilde{\tau}) = (x-y)\cdot\xi - (x-z)\cdot\tilde{\xi} + (t - y\cdot\omega)\tau - (t - z\cdot\omega)\tilde{\tau} + \frac{\sigma}{2\tau^2}(\tau^2 - |\xi|^2) - \frac{\tilde{\sigma}}{2\tilde{\tau}^2}(\tilde{\tau}^2 - |\tilde{\xi}|^2).$$

Since $d_x\phi = \xi - \tilde{\xi}, d_t\phi = \tau - \tilde{\tau}$, we can integrate by parts in $x, \tilde{\tau}$ to obtain a gain of $(1+|\xi-\tilde{\xi}|)^{-N}(1+|\tau-\tilde{\tau}|)^{-N}$ in the integrand. Then integrating in $x, t, \tilde{\xi}, \tau$, we obtain a new oscillatory representation of K_{A^*A} with phase

$$(z-y)\cdot(\xi+\tau\omega) + \frac{\sigma-\tilde{\sigma}}{2\tau^2}(\tau^2 - |\xi|^2),$$

amplitude $b_2(*;(\xi,\tau,\tilde{\sigma}))b_1(*;(\xi,\tau,\sigma))$, with $b_j \in S^{\mu-1}_{\frac{1}{2}}, j = 1,2$, and integrated with respect to $d\omega d\sigma d\tilde{\sigma} d\tau d\xi$. We can thus write

(2.17)

$$K_{A^*A}(z,y) = \int e^{i[(z-y)\cdot(\xi+\tau\omega)+\frac{\sigma(\tau^2-|\xi|^2)}{2\tau^2}]}b_1\tilde{*}b_2(*;\xi,\tau,\sigma)d\omega d\sigma d\tau d\xi,$$

where $b_1\tilde{*}b_2$ is the partial convolution in the σ variable. Using Proposition 2.2 with $k=1$, one has supp $(a_j) \subset \{< \sigma > \leq c < \xi, \tau >\}$, so that $b_{2\mu-1} = b_1\tilde{*}b_2 \in S^{2\mu-1}_{(1,0),(\frac{1}{2},0)}$ if $\mu - 1 < -1$, i.e., $\mu < 0$, with supp $(b_{2\mu-1}) \subset \{< \sigma > \leq c < \xi, \tau >, < \xi > \simeq < \tau >\}$. Letting $\eta = \tau\omega, d\tau d\omega = |\eta|^{1-n}d\eta$, and $\zeta = \xi + \eta$, we have

$$K_{A^*A}(z,y) = \int e^{i(z-y)\cdot\zeta}c(z,y;\zeta)d\zeta, \tag{2.18}$$

where

(2.19)

$$c(z,y;\zeta) = \int_{\mathbb{R}^{n+1}} e^{i\frac{\sigma}{2|\eta|^2}(2\eta-\zeta)\cdot\zeta}b_{2\mu-1}(z,y;\zeta-\eta,|\eta|,\sigma)|\eta|^{1-n}d\eta d\sigma.$$

We will show that $c \in S^{2\mu}_{0,0}$ if $-\frac{5}{4} < \mu < 0$.

We note that in the expression for $c(\zeta)$, the integration is over a region of the form

$$H_\gamma(\zeta) = \{\eta : \gamma^{-1}|\eta| \leq |\zeta - \eta| \leq \gamma|\eta|\}$$

for some $\gamma > 1$. This is $\mathbb{R}^n$ with two balls deleted:

$$H_\gamma(\zeta) = \mathbb{R}^n \backslash (B(-\delta\zeta; \tilde{\delta}|\zeta|) \cup B((1+\delta)\zeta; \tilde{\delta}|\zeta|)),$$

for appropriate $\tilde{\delta} > \delta > 0$ depending on γ. To estimate $c(y, z; \zeta)$, we divide $H_\gamma(\zeta)$ into five subregions, described below, noting that one may introduce a subordinate partition of unity which will not affect the $(1, 0), (\frac{1}{2}, 0)$ type of the amplitude that was described above.

Denote by ψ the phase function of (2.19), and note that

$$d_\eta \psi = \sigma \frac{|\eta|^2(\zeta - \eta) - |\zeta - \eta|^2 \eta}{|\eta|^4} \tag{2.20}$$

and

$$|d_\sigma \psi| = |\frac{(2\eta - \zeta) \cdot \zeta}{2|\eta|^2}| = \frac{|\zeta|}{|\eta|^2} \cdot dist(\eta, P_\zeta), \tag{2.21}$$

where P_ζ is the affine hyperplane $\frac{\zeta}{2} + \zeta^\perp$.

Region (I): $\{(\eta, \sigma) : \text{dist}(\eta, P_\zeta) \leq c|\zeta|, |\sigma| \leq c|\zeta|\}$.

We are working near the nonisolated critical points $\eta = \frac{\zeta}{2}, |\sigma| \leq c|\zeta|$. One calculates that $d^2\psi$ has rank 2 at these points, with a lower bound on a 2×2 minor being $c|\zeta|^{-2}$ uniformly. Performing stationary phase in two variables and integrating over a ball of radius $c|\zeta|$ in the remaining $n-1$, we obtain a contribution to $c(z, y; \zeta)$ dominated by

$$|\zeta|^{2\mu-1}|\zeta|^{1-n}(|\zeta|^{-2})^{-\frac{1}{2}}|\zeta|^{n-1} \leq c|\zeta|^{2\mu}.$$

Now pick an $\epsilon \in (0, \min(-\frac{1}{4\mu}, \mu + \frac{5}{4}))$. We form

Region (II): $\{(\eta, \sigma) : \eta \in H_\gamma(\zeta), \text{dist}(\eta, P_\zeta) \geq c\, max(\frac{|\eta|^{\frac{3}{2}+\epsilon}}{|\zeta|}, |\zeta|)\}$.

For such points, we have $|d_\sigma \psi| \geq c|\eta|^{\epsilon - \frac{1}{2}}$. Integrating by parts N times in σ yields an integrand which is $\leq |\eta|^{-2-n+N(\frac{1}{2}-\epsilon-\frac{1}{2})} \leq c|\eta|^{-2-n-\epsilon N}$. Integrating over $\{|\sigma| \leq c|\eta|\}$ and then integrating over $\{|\eta| \geq c|\zeta|\}$ yields a contribution to $c(y, z; \zeta)$ which is $O(|\zeta|^{-1-\epsilon N}) = o(|\zeta|^{2\mu})$ for $N > 1/\epsilon$.

Region (III): $\{(\eta, \sigma) : \eta \in H_\gamma(\zeta), c|\zeta| \leq |\eta| \leq c|\zeta|^{1/\epsilon}, \text{dist}(\eta, P_\zeta) \leq max(\frac{|\eta|^{\frac{3}{2}+\epsilon}}{|\zeta|}, |\zeta|)\}$.

We further decompose this into (IIIa): $\{c|\eta|^{\frac{1}{2}} \leq |\sigma| \leq \frac{|\eta|^{1+\epsilon}}{|\zeta|}\}$ and (IIIb): $\{\frac{|\eta|^{1+\epsilon}}{|\zeta|} \leq |\sigma| \leq c|\eta|\}$. On (IIIb), one has

$$|d_\eta \psi| \geq c\frac{|\sigma|}{|\eta|^2}|\eta - \frac{\zeta}{2}| \geq c\frac{|\eta|^\epsilon}{|\zeta|},$$

which allows us to integrate by parts N times in η. Since each differentiation in η gains $|\eta|^{-1}$, the integrand is $\leq c|\eta|^{-2-n}(\frac{|\zeta|}{|\eta|^{\epsilon}}|\eta|^{-1})^N$. Integrating in σ and then in η and introducing polar coordinates along P_ζ, this contribution to $c(y,z;\zeta)$ is dominated by

$$|\zeta|^N \int_{|\zeta|}^{|\zeta|^{\frac{1}{\epsilon}}} r^{-1-n-(1+\epsilon)N} r^{n-2}(r^{\frac{3}{2}+\epsilon}/|\zeta|)dr,$$

which is $\leq c|\zeta|^{-1-\epsilon N} + c|\zeta|^{\epsilon-\frac{3}{2}-\epsilon N} = o(|\zeta|^{2\mu})$ for N large. On (IIIa), integration by parts doesn't help and so we simply estimate the integral $d\sigma$ by $c|\eta|^{-2-n}|\eta|^{1+\epsilon}|\zeta|^{-1}$, which, when integrated in η is dominated by

$$\int_{|\zeta|}^{|\zeta|^{\frac{1}{\epsilon}}} \frac{r^{\epsilon-1-n}}{|\zeta|} r^{n-2} \frac{r^{\frac{3}{2}+\epsilon}}{|\zeta|} dr \leq c|\zeta|^{2\epsilon-\frac{5}{2}} = o(|\zeta|^{2\mu})$$

since $\epsilon < \mu + \frac{5}{4}$.

Finally, we have

Region (IV): $\{(\eta,\sigma) : \eta \in H_\gamma(\zeta), \ |\eta| \geq c|\zeta|^{\frac{1}{\epsilon}}, \ \mathrm{dist}(\eta, P_\zeta) \leq c\frac{|\eta|^{\frac{3}{2}+\epsilon}}{|\zeta|}\}$.

Here, we simply estimate the contribution to $c(y,z;\zeta)$ by

$$\int\int_{(IV)} |\eta|^{-2-n} d\sigma d\eta \leq c \int_{|\zeta|^{\frac{1}{\epsilon}}}^{\infty} r^{-1-n} r^{n-2} \frac{r^{\frac{3}{2}+\epsilon}}{|\zeta|} d,r$$

which is $\leq c|\zeta|^{-\frac{1}{2\epsilon}} = o(|\zeta|^{2\mu})$ if $\epsilon < -\frac{1}{4\mu}$.

Thus, we have shown that $|c(y,z;\zeta)| \leq c|\zeta|^{2\mu}$. Differentiation of the phase in ζ just multiplies the integrand by a function homogeneous of degree zero, which leaves the type and order of the amplitude unchanged, as does differentiation in the spatial variables. Hence, $c \in S^{2\mu}_{0,0}$, and thus $A^*A : L^2_{\mu,\mathrm{comp}} \to L^2_{-\mu,\mathrm{loc}} \Longrightarrow A : L^2_{\mu,\mathrm{comp}} \to L^2_{\mathrm{loc}}$; for $A = U_2$ we have $\mu = -1$ and thus $A : L^2_{-1,\mathrm{comp}} \to L^2_{\mathrm{loc}}$. Combined with the estimates for U_1, this gives Thm. 1(a) for $s < -\frac{1}{2}$. To obtain $U_2 : L^2_{-\frac{1}{2},\mathrm{comp}} \to B^{\frac{1}{2}}_{2,\infty,\mathrm{loc}}$, we need the following the analogue of Proposition 2.4, which exhibits the additional $\frac{1}{2}$ in the order of the singularity on $\{t = x\cdot\omega\}$ (in comparison with Proposition 2.4) mentioned at the end of the discussion of U_1.

Proposition 2.5. *Let $A \in I^{\mu-\frac{n}{4}}_\rho(\Lambda_2)$, and $B \in \Psi^{r_1}_\rho(\mathbb{R}^n)$, $C \in \Psi^{r_2}_\rho(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1})$ be properly supported. Then the Schwartz kernel*

$$K_{CAB} \in I^{\mu+r_1+r_2-\frac{n}{4}}_\rho(\Lambda_2) + C^\infty(\mathbb{R}^n; I^{\mu+r_2-\frac{n-1}{2}}_\rho(N^*\{t = x\cdot\omega\})).$$

Proof. Since $\pi_L : \Lambda_2 \to T^*(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1})\backslash 0$, as in the proof of Proposition 2.4 it suffices to deal with the case $C = I$. We make use of the alternative phase function

$$\tilde{\phi}(x, t, \omega, y; \xi, \tau, \sigma) = (x - y) \cdot \xi + (t - x \cdot \omega)\tau + \frac{\sigma}{2\tau^2}(\tau^2 - |\xi - \tau\omega|^2),$$

which also parametrizes Λ_2. Then we have the representation

$$K_A(x, t, \omega, y) = \int e^{i\tilde{\phi}} a_{\mu-1} d\xi d\tau d\sigma, \quad a_{\mu-1} \in S_\rho^{\mu-1},$$

and we may assume that $\frac{<\xi-\tau\omega>}{<\tau>} \in (\frac{2}{3}, \frac{3}{2})$ and $< \xi, \tau >^{1/2} \leq < \sigma >$ on the support of $a_{\mu-1}(x, t, \omega, y; \xi, \tau; \sigma)$. With $b_{r_1} \in S_\rho^{r_1}$ the amplitude of B, we have

(2.22)

$$K_{AB}(x, t, \omega, y) = \int e^{i[(x-y)\cdot\xi + (t - x\cdot\omega)\tau + \frac{\sigma}{2\tau^2}(\tau^2 - |\xi - \tau\omega|^2) + (y-z)\cdot\eta]} \cdot a_{\mu-1}(*; \xi, \tau; \sigma) \cdot b_{r_1}(*; \eta) dy d\xi d\tau d\sigma d\eta.$$

The proof now follows in general terms that of Proposition 2.4. Where $\frac{2}{3} < \frac{<\xi>}{<\eta>} < \frac{3}{2}$, stationary phase in y, η yields an oscillatory integral with phase $\tilde{\phi}$ and amplitude in $S_\rho^{\mu+r_1-1}$, yielding an element of $I_\rho^{\mu+r_1-\frac{n}{4}}(\Lambda_2)$. Since the y-gradient of the phase in (2.22) is $-\xi + \eta$, on the regions $\{< \xi > \geq \frac{3}{2} < \eta > + \frac{1}{2} < \sigma >\}$ and $\{< \eta > \geq \frac{3}{2} < \xi > + \frac{1}{2} < \sigma >\}$ one may integrate by parts any number M times in y, yielding an amplitude of arbitrarily negative order in ξ, η, σ. Where $max(< \xi >, < \eta >) \geq \frac{1}{2} < \tau >$, this may be integrated in all the phase variables to yield a C^∞ kernel; where τ is the elliptic variable, integrating in all variables except τ gives an expression

$$\int e^{i(t-x\cdot\omega)\tau} \tilde{a}_{\mu-1}(*; \tau) d\tau \in C^\infty(\mathbb{R}^n; I_\rho^{\mu-\frac{n+1}{2}}(N^*\{t = x \cdot \omega\})).$$

Finally, one must deal with the regions $\{\frac{1}{2} < \sigma > \geq < \xi > \geq \frac{3}{2} < \eta >\}$ and $\{\frac{1}{2} < \sigma > \geq < \eta > \geq \frac{3}{2} < \xi >\}$. Note that integrating in σ yields, by Proposition 2.3(b), the representation

$$\int e^{i[(x-y)\cdot\xi + (t-x\cdot\omega)\tau + (y-z)\cdot\eta]} a_\mu\left(*; \xi, \tau; \frac{\tau^2 - |\xi - \tau\omega|^2}{\tau^2}\right) b_{r_1}(*; \eta) dy s \eta d\xi d\tau,$$

with $a_\mu(*; \xi, \tau; s)$ of order μ in (ξ, τ), taking values in $I^{\mu-1}(\{s = 0\})$. If $< \xi > \geq \frac{3}{2} < \eta >$, then integration by parts M times in y and then integration in y, η gives

$$\int e^{i[x\cdot\xi + (t-x\cdot\omega)\tau]} a_\mu b_{-M'}(*; \xi) d\xi d\tau.$$

We may now integrate in ξ, noting that for each fixed τ, the integral in ξ is transverse to the singularity of a_μ at $\{|\xi - \tau\omega| = |\tau|\}$ and yields a (smooth) symbol in τ, giving $\int e^{i(t-x\cdot\omega)\tau} b_\mu(*;\tau)d\tau \in I^{\mu-\frac{n-1}{2}}(N^*\{t = x\cdot\omega\})$. On the other hand, if $<\eta> \geq \frac{3}{2} <\xi>$, we integrate by parts in y and then integrate in y, ξ to obtain

$$\int e^{i[(t-x\cdot\omega)\tau - z\cdot\eta]} b_\mu(*;\tau)c_{-M'}(*;\eta)d\tau d\eta = \int e^{i(t-x\cdot\omega)\tau}\tilde{b}_\mu(*;\tau)d\tau$$
$$\in C^\infty(\mathbb{R}^n; I^{\mu-\frac{n-1}{2}}(N^*\{t = x\cdot\omega\})),$$

finishing the proof of Proposition 2.5.

Now repeat the reasoning below (2.15), noting that (2.14) has been replaced by

(2.23)
$$K_{<D>^r U_2 <D>^{-r}} \in I_{\frac{1}{2}}^{-\frac{n+4}{4}}(\Lambda_2) + C^\infty(\mathbb{R}^n, I^{-\frac{n+1}{2}+r}(N^*\{t - x\cdot\omega = 0\})),$$

so that

$$U_2 : L^2_{r-1} \to L^2_r + I^{-\frac{n+1}{2}}(N^*\{t - x\cdot\omega = 0\}) \hookrightarrow L^2_r + B^{\frac{1}{2}}_{2,\infty,\mathrm{loc}}.$$

Hence, $U_2 : L^2_{s,\mathrm{comp}} \to L^2_{s+1,\mathrm{loc}}, \forall s < -\frac{1}{2}$ and $U_2 : L^2_{-\frac{1}{2},\mathrm{comp}} \to L^2_{\frac{1}{2},\mathrm{loc}} + B^{\frac{1}{2}}_{2,\infty,\mathrm{loc}} \hookrightarrow L^2_{\frac{1}{2},\mathrm{loc}}$, finishing the proof of Thm. 1(a).

3. $L^p_s \to L^{p'}_t$ estimates

We embed the operator U in an analytic family $U^\alpha, \alpha \in \mathbb{C}$, by inserting a factor $<\xi,\tau>^{-\alpha}$ into the amplitude of the oscillatory representation (2.3). Using the same parabolic cutoff as previously, decompose $U^\alpha = U^\alpha_1 + U^\alpha_2$, with $U^\alpha_j \in I_{\frac{1}{2}}^{-Re(\alpha)-1-\frac{n}{4}}(\Lambda_j), j = 1, 2$. By Proposition 2.1, we have that $U_1^{\alpha^*} U^\alpha_1 \in \Psi_{\frac{1}{2}}^{-2Re(\alpha)-2}(\mathbb{R}^n)$ for $-Re(\alpha) - 1 < -1/2$ and thus

$$U^\alpha_1 : L^2_{-Re(\alpha)-1,\mathrm{comp}} \to L^2_{\mathrm{loc}}, \quad -\frac{1}{2} < Re(\alpha). \tag{3.1}$$

Furthermore,

$$K_{U^\alpha_1}(x,t,\omega,y) = \int\int e^{i[(x-y)\cdot\xi + (t-x\cdot\omega)\tau]} b^\alpha_1 d\xi d\tau,$$

with $b_1^\alpha \in S_{\frac{1}{2}}^{-\frac{3}{2}-\mathrm{Re}\,(\alpha)}$. Thus, $K_{U_1^\alpha} \in L^\infty_{\mathrm{loc}}$ if $\mathrm{Re}\,(\alpha) > n - \frac{1}{2}$, and hence

$$U_1^\alpha : L^1_{\mathrm{comp}} \to L^\infty_{\mathrm{loc}}, \quad \mathrm{Re}\,(\alpha) > n - \frac{1}{2}, \tag{3.2}$$

Similarly, by the results of §2 for $I_{\frac{1}{2}}^{\mu-\frac{n}{4}}(\Lambda_2)$, we have

$$U_2^\alpha : L^2_{-Re(\alpha)-1,\mathrm{comp}} \to L^2_{\mathrm{loc}}, \quad -1 < Re(\alpha) < \frac{1}{4}, \tag{3.3}$$

and also $K_{U_2^\alpha} \in L^\infty_{\mathrm{loc}}$ if $\mathrm{Re}\,(\alpha) > n$, so that

$$U_2^\alpha : L^1_{\mathrm{comp}} \to L^\infty_{\mathrm{loc}}, \quad \mathrm{Re}\,(\alpha) > n. \tag{3.4}$$

It is straight forward to verify that the growth of the bounds in (3.1)-(3.4) are at most exponential in $|\mathrm{Im}\,(\alpha)|$. Interpolating (3.1) for $Re(\alpha) = -\frac{1}{2+\epsilon}$ with (3.2) for $Re(\alpha) = n - \frac{1}{2+\epsilon}$ for $\epsilon > 0$ arbitrarily small, we obtain $U_1^0 : L^p_{-\frac{1}{p'}} \to L^{p'}, \quad \forall p, \quad \frac{4n}{2n+1} < p < 2$. Interpolating (3.3) for $Re(\alpha) = -\frac{n-1}{2n}$ with (3.4) for $Re(\alpha) = n + \epsilon$, we obtain the same bounds for U_2^0 ; since $U = U_1^0 + U_2^0$, we have shown that

$$U : L^p_{-\frac{1}{p'},\mathrm{comp}} \to L^{p'}_{\mathrm{loc}}, \quad \frac{4n}{2n+1} < p < 2.$$

To extend this to $L^p_s \to L^{p'}_t$ estimates, we use

Proposition 3.1. *If $T \in I_\rho^{\mu-\frac{n}{4}}(\Lambda_j)$, $j = 1$ or 2, is continous $L^p_{s,\mathrm{comp}} \to L^q_{t,\mathrm{loc}}$ for some $s, t \in \mathbb{R}$, then*

$$T : L^p_{s+r,\mathrm{comp}} \to L^q_{t+r,\mathrm{loc}} \quad \forall r < \frac{1}{q} - \mu - t - 1.$$

Proof. If $A_j \in I_\rho^{\mu-\frac{n}{4}}(\Lambda_j)$ and $B \in \Psi^{-r}(\mathbb{R}^n), C \in \Psi^r(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1})$ are elliptic and properly supported with parametrices B^{-1}, C^{-1}, then

$$K_{CA_jB} \in I_\rho^{\mu-\frac{n}{4}}(\Lambda_j) + C^\infty(\mathbb{R}^{n+1} \times \mathbb{S}^{n-1}; I^{\mu+r-\frac{n-1}{2}}(N^*\{t = x\cdot\omega\}))$$

(or better) by Proposition 2.4 and 2.5. An element of the first space maps $L^p_{s,\mathrm{comp}} \to L^q_{t,\mathrm{loc}}$ by assumption and therefore its contribution to $A_j = C^{-1}(CA_jB)B^{-1}$ maps $L^p_{s+r,\mathrm{comp}} \to L^q_{t+r,\mathrm{loc}}$. For the second term, we note that operators with Schwartz kernels in $C^\infty(\mathbb{R}^{n+1}\times\mathbb{S}^{n-1}; I^{\mu+r-\frac{n-1}{2}}(N^*\{t=x\cdot\omega\}))$ map $L^p_{s,\text{comp}}$ to $I^{\mu+r-\frac{n-1}{2}}(N^*\{t=x\cdot\omega\})$. The inclusion

$$I^{\mu+r-\frac{n-1}{2}}(N^*\{t=x\cdot\omega\}) \hookrightarrow L^q_{t,\text{loc}}$$

holds if $q<(\mu+r+t+1)^{-1}$. Thus, if $r<min(-\mu-t,\frac{1}{q}-\mu-t-1)=\frac{1}{q}-\mu-t-1$, then the second contribution to A_j maps $L^p_{s+r,\text{comp}}\to L^q_{t+r,\text{loc}}$. □

Applying the Proposition to U, we obtain

$$U: L^p_s(\mathbb{R}^n)\to L^{p'}_{s+\frac{1}{p'}}(\mathbb{R}^{n+1}\times\mathbb{S}^{n-1}), \quad s<0, \quad \frac{4n}{2n+1}<p<2,$$

yielding Thm. 1(b).

References

1. A. P. Calderón and R. Vaillancourt, A class of bounded pseudodifferential operators, Proc. Nat. Acad. Sci. USA **69** (1972) 1185–1187
2. H. Federer, Geometric Measure Theory, Springer-Verlag 1969
3. A. Greenleaf, G. Uhlmann, Estimates for singular Radon transforms and pseudo-differential operators with singular symbols, J. Funct. Anal. **89** (1990) 202–232
4. A. Greenleaf, G. Uhlmann, Recovering singularities of a potential from singularities of scattering data, Comm. Math. Phys. **157** (1993) 549–572
5. V. Guillemin and S. Sternberg, Geometric Asymptotics, Amer. Math. Soc. 1977
6. V. Guillemin and G. Uhlmann, Oscillatory integrals with singular symbols, Duke Math. J. **48** (1981) 251–267
7. L. Hörmander, The Analysis of Linear Partial Differential Operators, vols. 1–4 Springer-Verlag 1984
8. A. Melin, The Lippmann-Schwinger equation treated as a characteristic Cauchy problem, Sem. sur les equ. aus der. par. (Ec. poly.) **12** (1988-1989)
9. A. Melin, On the use of intertwining operators in inverse scattering, in Schrödinger operators (Sonderborg, 1988) ed. by H. Holden and A. Jensen, Lecture Notes in Phys. 345 Springer-Verlag 1989
10. R. Melrose, Notes on marked lagrangians, M.I.T. lecture notes
11. R. Melrose and G. Uhlmann, Lagrangian intersection and the Cauchy problem, Comm. Pure Appl. Math. **32** (1979) 482–519
12. G. Mendoza, Symbol calculus associated with intersecting lagrangians, Comm. PDE **7** (1982) 1035–1116
13. P. Stefanov, A uniqueness result for the inverse back-scattering problem, Inverse problems **6** (1990) 1055–1064
14. R. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, Duke Math. Jour. **44** (1977) 705–713
15. H. Triebel, Theory of Function Spaces, Birkhäuser 1983